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Numerical analysis of a tag model in circle

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We analyzed a simplest tag model on a circle. This problem consists of one person to chase and one to elude, and the output force to move is a function of the relative position. The effect of time delay from collecting the information to output is considered. This model shows various motion including chaotic one, which can not be observed without the time delay. When replacing the delay term to a distributed one, some chaotic motion is stabilized.

I. INTRODUCTION

Chasing is ubiquitous around us. Tag is a typical children's game in which one chases the rest. In a ball game such as soccer of basketball, a person possessing ball are chased by the defense. The dynamics of the person to chase("chaser") and the person to elude("eluder") is not trivial: the chaser collects the information of the eluder (position, velocity, etc.), process the information, and determine the amount and the direction of the output force, and vice versa. In general, processing information costs a finite time. In this study, we focus on the effect of the time delay to process the information.

Time delay in a dynamical system is common in nature. A typical example is the maturation time of man to reproduce the next generation. It takes a finite time that a disturbance of birth rate is reflected in a next generation. When this effect is taken in a population model, it contains delay term[1].

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Time delay appears in many kinds of dynamical system such as demography (maturation time), epidemic (incubation time), control systems (transmission of feedback signal), economy (time lag from the measurement to the announce of the economic indicators), optics (feedback signal of light) [1, 3, 4]. It is known that time delay destabilize a fixed point and invokes complex phenomena such as chaos.

In this paper, we make a simplest tag model to understand the chasing dynamics. Our aim here is to know how the time delay changes the result of chasing without delay. In particular, we assume that the chaser and the eluder have similar moving principle. Furthermore, we choose the maximum of the output function as a parameter of the ability to move. The difference of the parameter between the chaser and the eluder is an indicator to predict the result of the chase. An naive expectation is that the sign of the indicator is a unique factor to determine the result. On the other hand, time delay causes to destabilize the fixed point. These two elements may be in conflict, so the behavior of the tag problem is not trivial.

In sec.II, we show the detail of the model. Sec. III is devoted to show the numerical result of the model. Linear stability theory is applied to this model in sec. IV.

II. MODEL

We consider tag model in a unit circle with one chaser(X) and one eluder(Y). Each position of X and Y, x and y, is measured by the arc length from an origin. The dynamics of X and Y is given by the following equations:

$$\ddot{x}(t) + A\,F(z(t-\tau)) + K\dot{x}(t) = 0, \quad (1)$$
$$\ddot{y}(t) + B\,F(z(t-\tau)) + K\dot{y}(t) = 0, \quad (2)$$
$$z(t) = y(t) - x(t) \quad (3)$$

where $\dot{x}$ means time derivative of x, A, B, K are positive constants. The output force is determined by $F(z)$ as a function of the relative position of X and Y, $z = z(t)$. To realize the situation in tag, $F(z)$ should be positive when $0 < z < \pi$, negative when $\pi < z < 2\pi$. The output-force function $F(z)$ is assumed odd by the requirement of the symmetry under space inversion, and it is also assumed $2\pi-$periodic function because of the periodicity of space. Here we assume finite processing time τ from the input of the information to output the force. In this model, the information is the relative position of X and Y.

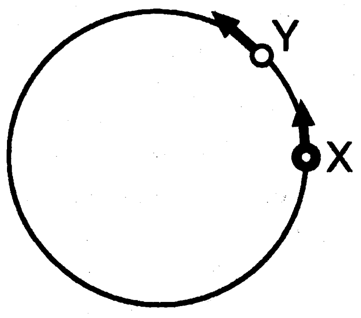

FIG. 1: Schematic picture of the tag model. X and Y move in the unit circle. X chases Y, while Y eludes from X.

This equation can be rewritten in terms of $z(t)$ only as follows:

$$\ddot{z}(t) + C\,F(z(t-\tau)) + K\dot{z}(t) = 0, \quad (4)$$

where $C = A-B$ is the parameter to measure the maximum of the output force ($C > 0$ means X(chaser) has larger output force than Y(eluder)).

The definition of the capture is as follows: in this simple situation, a naive definition of the catch, such as the state $|z(t)| < \exists\epsilon$, for $\exists t$ can not lead us to interesting results. Thus we define the catch as the state $z(t) \to 0$(also $\dot{z}(t) \to 0$) as $t \to \infty$. This definition means that catching Y requires not only the relative position but the relative velocity should be zero. A simplest example to satisfy this condition is tag among two agile persons. An instant time of coincidence is insufficient to catch the eluder: he can slip through from the chaser's arms.

In this paper, we report the case $F(x) = \sin(x)$, while the case of the other function such as bilinear function($F(x) = x(|x| < \pi/2); \pi - x(\pi/2 < x < 3\pi/2)$) and rectangular function($F(x) = 1(0 < x < \pi); -1(\pi < x < 2\pi)$) is also studied in ref. [5].

When $\tau = 0$, the equation (4) corresponds to a dumped simple pendulum. In this case, the behavior is simple. Equilibrium points are $z(t) = 0$ and $z(t) = \pi$. If $C > 0$ $z(t) = 0$ is stable and $z(t) = \pi$ is unstable, and if $C < 0$, vice versa. In terms of the tag model, these result is trivial. The case $C > 0(A > B)$ means the chaser's ability to acceleration is larger that the eluder. Thus it is reasonable that the stable equilibrium point is $z(t) = \dot{z}(t) = 0$, which means the state of the catch. On the other hand, the case $C < 0(A < B)$ means the eluder's ability to acceleration is larger that the chaser. Thus the eluder will not be caught by the chaser. Stable equilibrium point, $z(t) = \pi, \dot{z}(t) = 0$, means that the relative position is not zero.

In the next section, we study this model numerically in the case of $\tau > 0$. Because $\tau \neq 0$, we change the variable of time as $t \to$

τt, so that time delay τ is unit time. Under this transformation, we have

$$\ddot{z}\,(t) + c\,F(z(t-1)) + k\dot{z}(t) = 0, \quad (5)$$
$$c = C\tau^2 \quad , \quad (6)$$
$$k = K\tau \quad . \quad (7)$$

We analyze this equation hereafter.

III. NUMERICAL RESULT

Eq.(5) is integrated numerically by the first-order Euler method with time step $\Delta t = 0.005$. The initial condition is chosen randomly. Because eq.(5) is invariant under the transformation $c \longrightarrow -c, z \longrightarrow \pi - z$, we only calculate the case $c > 0$. Parameters c and k is changed in the range $0 < c < 20$ and $0 < k < 8$.

A. Phase Diagram

We analyzed the time variation of the relative position, $z(t)$, after transition time. The data is classified into the following four categories (fig.2): (I) convergence ($z(t) =$ const.), (II) simple periodic ($z(t)$ is a periodic function, and it has just one maximum in a period like sine function), (III) complex periodic ($z(t)$ is a periodic function, and it has more than one maximum in a period), (IV) non-periodic (which will be referred to as "chaotic" hereafter).

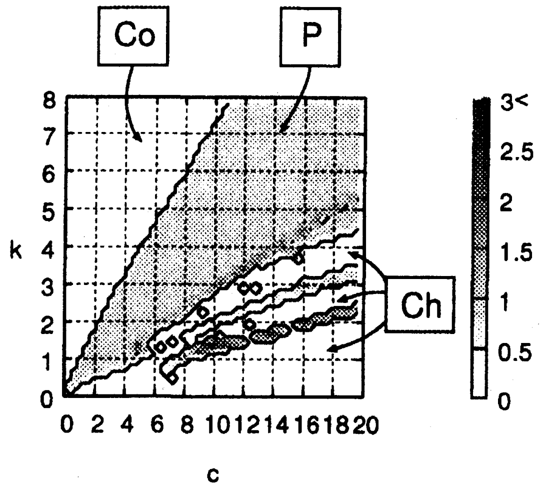

FIG. 2: Phase diagram of the tag model. For any set of (c, k), phase is automatically analyzed using $z(t)$, and is characterized by phase number. In the cases of (I) and (IV), phase number is 0. and in the cases of (II) and (III), phase number is period number if it is smaller than 3. When period number is larger than 3, phase number is 3. Boundary between the region where phase number is zero and nonzero is shown.

We have one stable steady solution $z(t) = 0$ in the region **Co**. This area is described by $k > 0.8c$ approximately, In this region, the dump term $k\dot{z}(t)$ surpasses the output-force term $cF(z(t-1))$. Eqs. (6) and (7) shows that another interpretation for this phase diagram. For example, the behavior of eq. (4) with a set of (C, K, τ) is equivalent to the behavior of eq. (5) with a set of $(c, k) = (C\tau^2, K\tau)$. Therefore, if one want to consider how the behavior of eq. (4) varies under the delay time τ with fixed C and K, the change of the behavior is shown along the parabola $k = \frac{K}{\sqrt{C}}\sqrt{c}$ in fig. 2.

We have a region of periodic solutions, **P**, indicated by roughly $0.8c > k \gtrsim 0.25c$. The gray color shows "the period number of periodic solution", which is the number of maxima in one period (note that this is half of the definition in ref. [2]). In the most of the region, the period number is 1: simple oscillation like a sine function. We have narrow region of complex periodic solution. This region is tangled, but one of the region is around $k \simeq 0.1c, c > 7$.

The period of the periodic solution is shown in fig.3. Roughly speaking, the period is determined mainly by k except near the boundary of this region. The region can be separated into two regions by the line $k = k_c \simeq 2.5$. In the area where $k < k_c$, the period is a rapid decreasing function of k, while when $k > k_c$, the rate of decreasing becomes small.

The period seems to converge to a certain value as a limit $k \to \infty$. This can be understood as follows. Periodic solution is achieved when $c \sim k$. So in the limit $k \to \infty$, the second term and the third term in the eq.(5) balance each other. It means that the phase of two terms should be coincides. Thus phase shift for $z(t)$ in the term $cF(z(t-1))$ is $\frac{2\pi}{T}$(T is the period of the solution), and in the term $k\dot{z}(t)$ is $\frac{\pi}{2}$. The balance of the two terms an asymptotic value of the period: $T = 4$. The detail of the analysis can not be written in this paper because of the page limitation, but we report it elsewhere.

In the region roughly $k < 0.25c$, we have three "chaotic" regions (**Ch**) which is separated narrow periodic regions ($k \simeq 0.15c$ and $k \simeq 0.1c, c > 7$). This region shows aperiodic solution. We discuss the behavior in this region in the following subsections mainly.

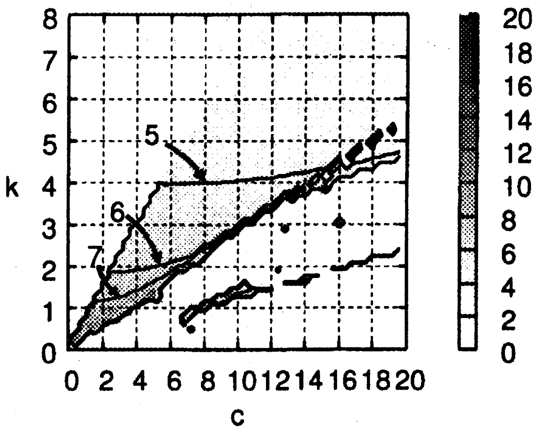

FIG. 3: Period of the periodic solution. The non-period region (convergence, chaotic) is shown by white. Period is shown by gray scale, and three contours (period= 5, 6, 7) are drawn for convenience.

B. Orbit in Phase Space

Typical orbits in the space spanned by $(z(t), z(t-1))$ are shown in fig.4. (we fixed $k = 2$, and c is changed).

Parameters for fig. 4(a) and fig.4(b) are in the region **P** in fig.2, 4(c) and 4(e) in the region **Ch**. Parameters in fig. 4(d) and fig. 4(f) are in a narrow bands in **Ch** region in fig.2.

A simple oscillation is shown in fig. 4(a), which is symmetric with respect to the unstable equilibrium point $z(t) = \pi$ (corresponding to $(z(t), z(t-1)) = (\pi, \pi)$). A complex periodic solution is shown in fig. 4(b). This solution is a result of symmetry-breaking, and the shape is not symmetric with respect to itself. In this region, we have two solutions symmetric with each other, which is bifurcated from a symmetric periodic solution. Fig.4(c) shows a chaotic solution. The chaotic solution seems to twist around an unstable periodic solution. When this solution is shown in $t - z$ space (not shown in this paper), it shows mostly an oscillation around $z(t) = 0$, but sometimes it shows a rotary motion. It should be noted that the this orbit passes near the origin, which is an unstable equilibrium point. The aperiodic motion seems to originate from the unstableness of the origin. However, there is a hole where the path of the orbit never passes. It indicates that the cross section of the orbit has a narrow width (see also fig.5 and fig. 6).

Another periodic orbit is shown in fig. 4(d). This solution shows a rotation in S^1, which is different from the oscillation in fig.4(a) and 4(b). This rotary periodic solution is sandwiched between two chaotic regions characterized by fig.4(c) and fig.4(e). Chaotic solution shown in fig.4(e) differs from the chaotic solution in fig.4(c) in the sense that the orbit is always rotary. The orbit covers most of the space, although the cover ratio is dependent on c.

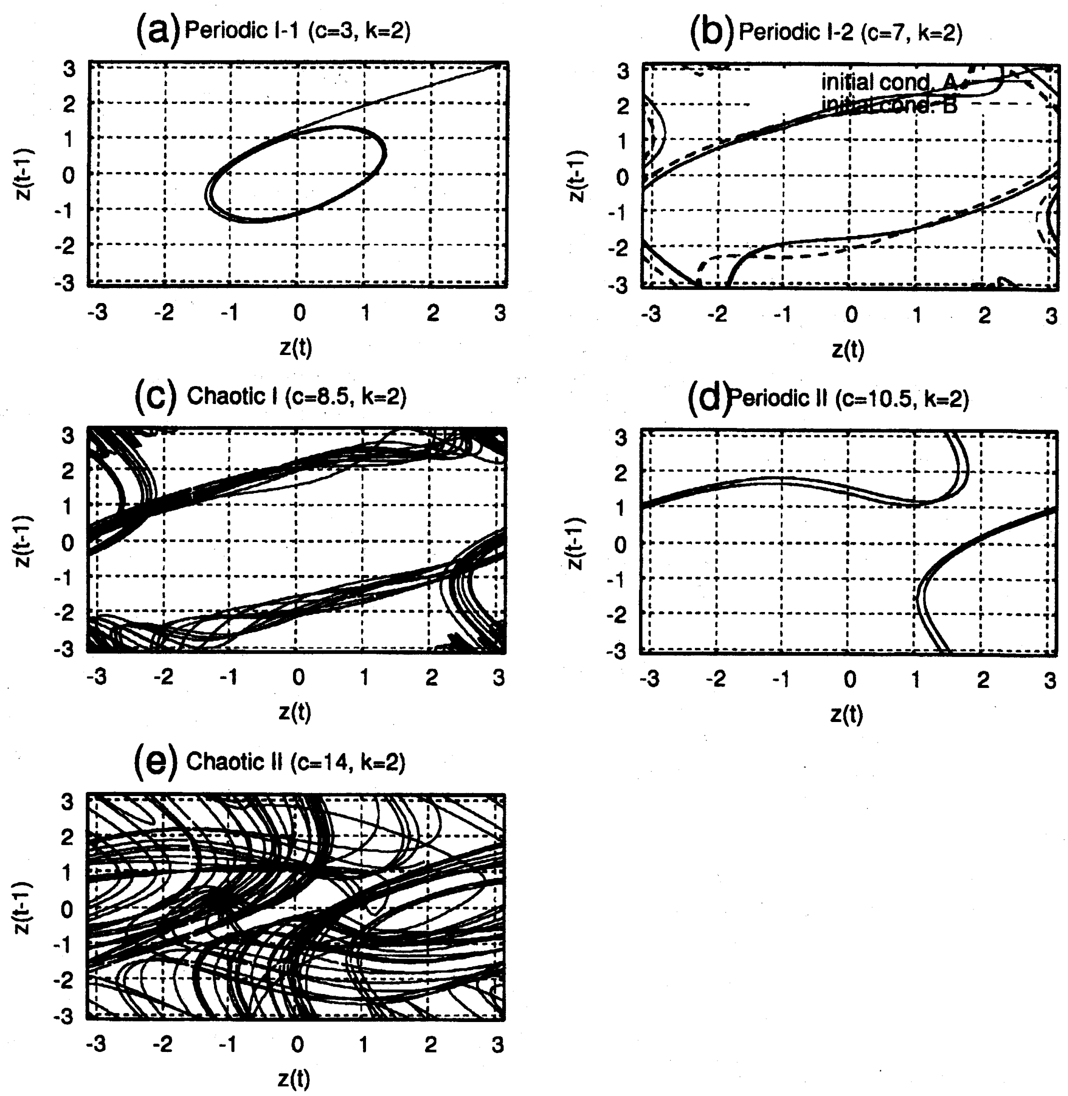

FIG. 4: Orbit in the phase space $(z(t), z(t-1))$. To display the orbit clearly, we draw two periods for each axis.

C. Cross Section

We show a Poincare section (plot of $z(t-1)$ when $z(t) = 0$ and $z(t) > 0$) of the orbit in the space spanned by $(z(t), z(t-1))$ in fig. 5, and fig. 6(magnification of fig.5). The value of k is fixed to 2, which is the same as fig.4. In fig. 6, a bifurcation from a symmetric periodic solution to asymmetric ones is shown around $c \simeq 6.8$. The transition to chaos is observed by $c \simeq 7.3$. In the region $7.3 < c < 10$, Poincare section has a width about $0.5 \sim 1.0$(depending on c), which corresponds to the situation typically shown by fig.4(c), although many window-like structure is observed in the region. In the region $c > 11$, the width increasing with c, and when $c > 15$, the section covers whole the domain of $z(t)$(except the window $16 < c < 17$).

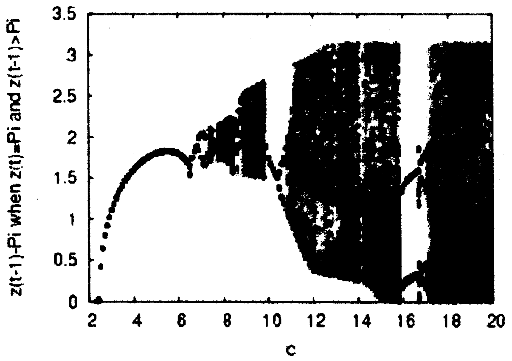

FIG. 5: Cross section of orbit.

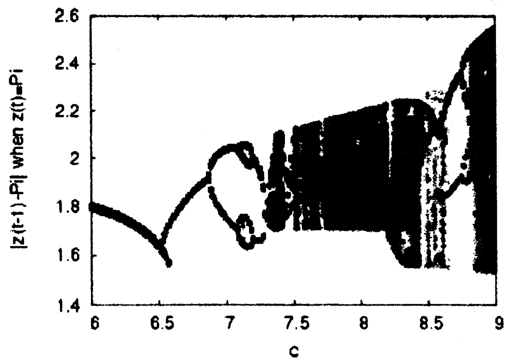

FIG. 6: The same as fig.5, but enlarged to show the transition to chaos.

D. Stabilization of Chaotic Behavior by Distributed Delay

In this subsection, we analyze the effect of the suppression of chaotic behavior by the distributed delay. Recently, Thiel et. al.[2] reported that delay-differential equation shows a kind of simplification of complex behavior when delay term is replaced by the distributed delay. In their paper, simplification means reduction of oscillation amplitude or period. In particular, their result shows chaotic behavior recovers periodicity as the width of the delay distribution increases.

We study a distributed delay version of eq.(5).

$$\ddot{z}(t) + c\,F(\bar{z}(t-1)) + k\dot{z}(t) = 0, \qquad (8)$$

$$\bar{z}(t) = \int_{-\infty}^{\infty} z(t-1-t')P(t')\mathrm{d}t', \qquad (9)$$

$$P(t) = \begin{cases} \frac{1}{\sigma} & (t \le \frac{\sigma}{2}) \\ 0 & (t > \frac{\sigma}{2}) \end{cases}, \qquad (10)$$

where σ is the width of the distribution, and distribution function $P(t)$ is a normalized uniform distribution in $[-\frac{\sigma}{2}, \frac{\sigma}{2}]$, $\int_{-\infty}^{\infty} P(t)\mathrm{d}t = 1$.

A detailed analysis shows ([5]) the chaotic region which is adjoint to **P**,($7.3 < c < 10$ when $k = 2$) vanishes as ρ increases, while other chaotic region remains chaotic. A typical stabilizing process for a parameter set $(c, k) = (14, 3.36)$ is shown in fig.7. Roughly speaking, stabilization seems to start when $\sigma > 1$. A branch which survives to a periodic solution when $\sigma > 1.2$ stems when $\sigma = 1$, and chaotic region starts inverse period-doubling cascading until it vanishes at $\sigma \simeq 1.2$.

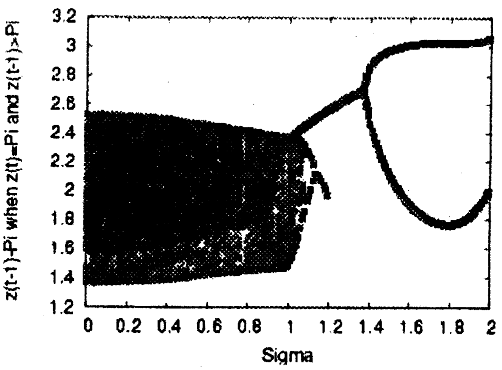

FIG. 7: Poincare section of the orbit ($c = 14, k = 3.36$), as a function of the width of the distribution σ.

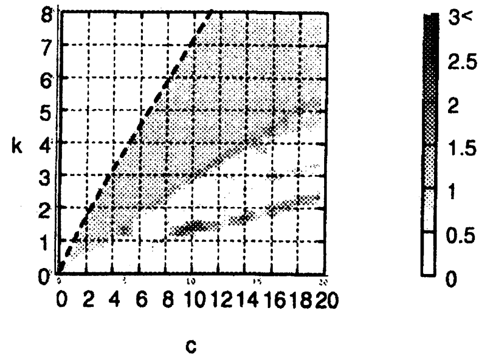

FIG. 8: The stability boundary of linear theory.

IV. THEORETICAL RESULT

A. linear stability

In this section, we check the agreement of linear stability theory with numerical results.

We start with a linearized equation:

$$\ddot{y}(t) + cy(t-1) + k\dot{y}(t) = 0. \tag{11}$$

It is easily known that the characteristic equation for eq.(11) is

$$\sigma^2 + ce^{-\sigma} + k\sigma = 0 \tag{12}$$

by putting $y(t) \propto e^{\sigma t}$.

To obtain stability boundary, we set $\sigma = i\omega(\omega \in \mathbf{R})$, and we get the equation for ω,

$$\begin{aligned} \omega^2 &= c\cos(\omega) && (13)\\ k\omega &= c\sin(\omega) && (14)\\ & && (15) \end{aligned}$$

We can get the equation for ω by calculating $(13)^2 + (14)^2$:

$$\omega^4 + k^2\omega^2 - c^2 = 0. \tag{16}$$

It is easily shown that the solution of eqs.(14) ω exists. The explicit form of ω is:

$$\omega = k\sqrt{\mu} \tag{17}$$

$$\mu = \frac{1}{2}\left(\sqrt{1+\left(\frac{2}{\eta}\right)^2} - 1\right), \tag{18}$$

where $\eta = k^2/c$.

Getting the equation of τ_c is as follows: Calculating $k^2(13) + (14)^2$ is

$$s^2 + \eta s - 1 = 0, \tag{19}$$

where $s = \cos(\omega)$. The solution of eq.(19) is

$$s = \cos(\omega) = \eta\mu. \tag{20}$$

We denote $\cos^{-1}(x)(= \theta)$ by the smallest non-negative value which satisfies $\cos(\theta) = x$ to obtain

$$\omega = \cos^{-1}(\eta\mu). \tag{21}$$

Equating eq. (18) to eq. (21), we get the equation to determine the stability boundary:

$$k\sqrt{\mu} = \cos^{-1}(\eta\mu) \tag{22}$$

Fig. 8 shows the superposed picture of fig.2 and stability boundary of eq. (22). Eq. (22) agrees well with the numerical result.

V. SUMMARY

Chasing problem among two object is analyzed on a circle using simple delay-differential equation. In the case without delay, this equation only shows the capture or the flee depending on the parameter determining the maximum output force. Unlike such trivial result above, this equation shows various motions when time delay of processing information is considered. If the time delay is considered, the tag does not reach to an end in a wide range of parameters. In such parameter, the eluder eludes from the chaser even if the ability of the chaser is larger than the eluder.

We analyzed the behavior numerically, and classified it into convergent state, periodic state, and chaotic state. Linear stability theory reproduces the boundary between the convergent state and the periodic state well. The effect of replacing delay term with distributed delay one is also studied, and it is shown that a part of chaotic region is stabilized. The reason is unknown, but in such stabilized region, Poincare map of the non-distributed equation has a confined region.

In the phase space spanned by C(the maximum of the output force) and K(dump coefficient), the boundary of these state are characterized roughly by the line $K \propto C$. A simple phenomenological theory is being constructed, and it seems to account for the reason, but we will report the detail for another opportunity.

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